Estimation under L-moment condition models

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27 August 2013

Plan

Definition

Sample : $X_1,\dots,X_n$ real random variable of common distribution function F

$$\lambda_r = \frac{1}{r}\sum_{k=0}^{r-1}(-1)^k\binom{r-1}{k}\mathbb{E}[X_{r-k:r}]$$

$X_{1:n} \leq X_{2:n} \leq \dots \leq X_{n:n}$: order statistics

- $\lambda_1 = \mathbb{E}[X]$: measure of location
- $\lambda_2 = \mathbb{E}[X_{2:2} - X_{1:2}]$: measure of scale
- $\frac{\lambda_3}{\lambda_2} = \frac{\mathbb{E}[X_{3:3}-2X_{2:3}+X_{1:3}]}{\mathbb{E}[X_{2:2}-X_{1:2}]}$: measure of skewness
- $\frac{\lambda_4}{\lambda_2} = \frac{\mathbb{E}[X_{4:4}-3X_{3:4}+3X_{2:4}-X_{1:4}]}{\mathbb{E}[X_{2:2}-X_{1:2}]}$: measure of kurtosis

Existence since $\int |x| dF(x) < \infty$

Definition (continuous distributions)

If F is continuous :

$$\mathbb{E}[X_{j:r}] = \frac{r!}{(j-1)!(r-j)!} \int_{\mathbb{R}} xF(x)^{j-1}(1-F(x))^{r-j} dF(x)$$

L-moments can then be written :

$$\lambda_r = \int_0^1 F^{-1}(t) L_r(t) dt = \int_0^1 F^{-1}(t) dK_r(t)$$

with

- L_r shifted Legendre polynomials (orthogonal basis in $L_2([0,1])$)

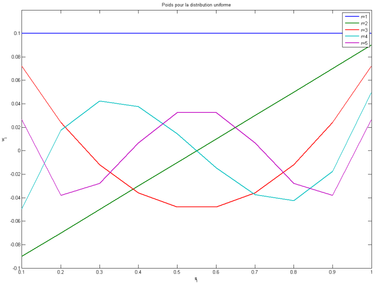
$$L_r(t) = \sum_{k=0}^{r} (-1)^k \binom{r}{k}^2 t^{r-k} (1-t)^k = \sum_{k=0}^{r} (-1)^{r-k} \binom{r}{k} \binom{r+k}{k} t^k$$

- $K_r(t) = \int_0^t L_{r-1}(u) du$

Definition (discrete distributions)

L-moments for a multinomial of support $x_1 \leq x_2 \leq \ldots \leq x_n$ and associated weights $\pi_1, \ldots, \pi_n$ ($\sum_{i=1}^{n} \pi_i = 1$)

$$\lambda_r = \sum_{i=1}^{n} w_i^{(r)} x_i = \sum_{i=1}^{n} \left[K_r \left(\sum_{a=1}^{i} \pi_a \right) - K_r \left(\sum_{a=1}^{i-1} \pi_a \right) \right] x_i$$



Estimation of L-moments

Unbiased estimator : U-statistics

$$l_r^{(u)} = \frac{1}{\binom{n}{r}} \sum_{1 \le i_1 < \cdots < i_r \le n} \frac{1}{r} \sum_{k=0}^{r-1} (-1)^k \binom{r-1}{k} x_{i_{r-k}:n}$$

Biased estimator : V-statistics (L-moments of the empirical distribution)

$$l_r = \frac{1}{\binom{r+n-1}{n-1}} \sum_{1 \le i_1 \le \cdots \le i_r \le n} \frac{1}{r} \sum_{k=0}^{r-1} (-1)^k \binom{r-1}{k} x_{i_{r-k}:n}$$

Illustration : asymptotic quality for a Gumbel family

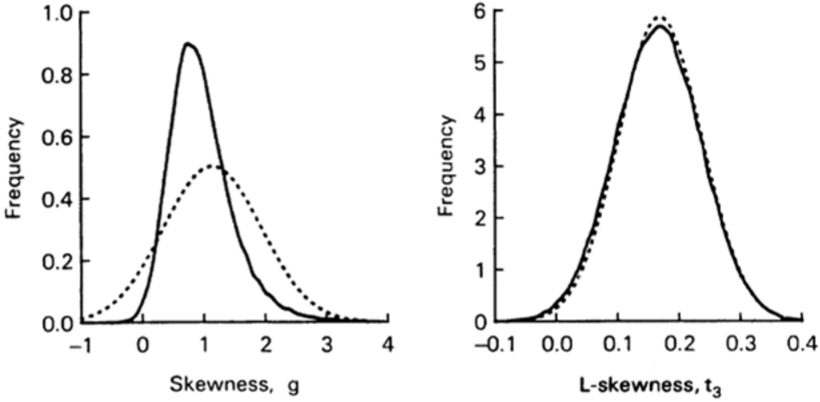
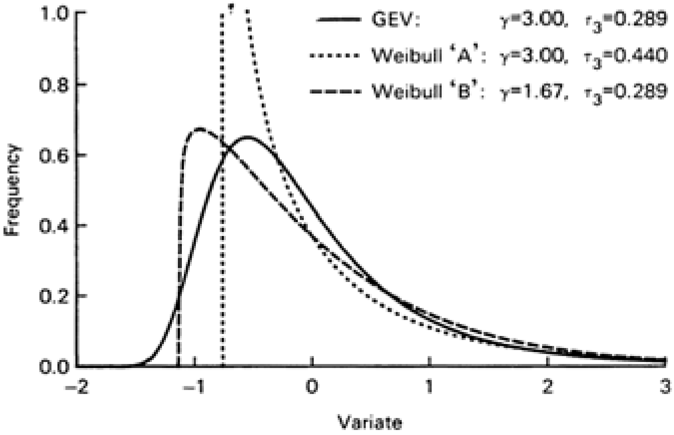


Illustration : preservation of shape information



Moment and L-moment equations models

- Semi-parametric model defined through moments

$$\int_{\mathbb{R}} g(x,\theta)dF(x) = \mathbb{E}[g(X,\theta)] = 0$$

where $g : \mathbb{R} \times \Theta \to \mathbb{R}^l$

- Semi-parametric model defined through L-moments

$$\mathbb{E}\left[\frac{1}{r}\sum_{k=0}^{r-1}(-1)^k \binom{r-1}{k} X_{k:n}\right] - \lambda_r(\theta) = 0 \quad 2 \leq r \leq l$$

equivalent to

$$\int_{\mathbb{R}} K(F(x))dx - f(\theta) = 0 \text{ with } f = (f_2, \ldots, f_l) \text{ and } K = (K_2, \ldots, K_l)$$

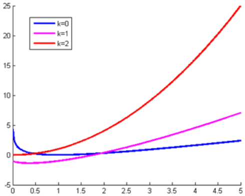
L-moment model example

L-moments of the Weibull family :

$$\begin{cases} \int_{\mathbb{R}} K_2(F(x))dx = f_2(\sigma,\nu) = -\sigma(1-2^{-1/\nu})\Gamma(1+1/\nu) \\ \int_{\mathbb{R}} K_3(F(x))dx = f_3(\sigma,\nu) = f_2(\sigma,\nu)[3-2\frac{1-3^{-1/\nu}}{1-2^{-1/\nu}})] \\ \int_{\mathbb{R}} K_4(F(x))dx = f_4(\sigma,\nu) = f_2(\sigma,\nu)[6+\frac{5(1-4^{-1/\nu})-10(1-3^{-1/\nu})}{1-2^{-1/\nu}})] \end{cases}$$

φ-divergence

Let $\varphi : \mathbb{R} \to [0, +\infty]$ be a strictly convex function with $\varphi(1) = 0$
For example : Cressie-Read family $\varphi(u) = \frac{u^k - ku + k - 1}{k(k-1)}$



Divergence between distributions F and G :

$$D_\varphi(F, G) = \int_{\mathbb{R}} \varphi\left(\frac{dG}{dF}(x)\right) dF(x)$$

where $\frac{dG}{dF}$ is the Radon-Nikodym derivative.

Minimum of φ-divergence

The model would be :

$$M_\theta^{(0)}(F) = \{G \in M_+ | G \ll F, \int_{\mathbb{R}} K(G(x))dx = f(\theta)\}$$

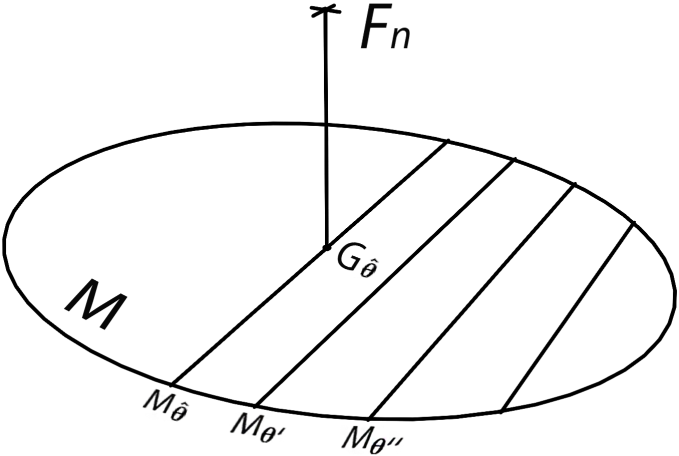
Let $F_n(x) = \sum_{i=1}^n \frac{1}{n} \mathbb{1}_{x > x_i}$ be the empirical distribution for a sample $x_1, \ldots, x_n$. The plug-in estimator is then :

$$\hat{\theta}_n^{(0)} = \arg \inf_{\theta \in \Theta} \inf_{G \in M_\theta^{(0)}(F_n)} D_\varphi(F_n, G)$$

But ;

- existence?
- quick computation?

Minimum of φ-divergence : illustration



Projection for the L-moment constraint model

We choose to minimize the divergence between quantile measure in order to obtain linearity
The model become

$$M_\theta(F) = \{G \in M_+ | G^{-1} \ll F^{-1}, \int_0^1 K(u) dG^{-1}(t) = f(\theta)\}$$

and the estimator

$$\hat{\theta}_n = \arg \inf_{\theta \in \Theta} \inf_{G \in M_\theta(F_n)} \int_0^1 \varphi\left(\frac{dG^{-1}}{dF_n^{-1}}\right) dF_n^{-1}(u)$$

Relation with transport

If now $T_n(x) = G^{-1} \circ F_n(x)$ for $x \in \mathbb{R}$, we rewrite the estimator

$$\inf_{\int_{\mathbb{R}} K(F_n(x))dT_n(x)=f(\theta)} \int_{\mathbb{R}} \varphi\left(T_n'(x)\right) dx =$$
$$\inf_{\int_0^1 K(u)dG^{-1}(u)=f(\theta)} \int_0^1 \varphi\left(\frac{dG^{-1}}{dF_n^{-1}}\right) dF_n^{-1}(u)$$

T_n is a transport between the random variable X of distribution function F_n and Y of distribution function G : $T_n(X) \stackrel{d}{=} Y$

Dual representation

Fenchel-Legendre transform of φ

$$\forall t \in \mathbb{R}, \quad \psi(t) = \sup_{x \in \mathbb{R}} \{tx - \varphi(x)\}$$

Proposition

Let $\theta \in \Theta$ and F be fixed.
If there exists some T in $M_\theta(F)$ such that $\frac{dT}{d\mu} \in int(dom(\varphi))$ μ-a.s.

$$\inf_{\int_{\mathbb{R}} K(F(x))dT(x)=f(\theta)} \int \varphi\left(\frac{dT}{d\mu}\right) d\mu$$
$$= \sup_{\xi \in \mathbb{R}^l} \xi^T f(\theta) - \int_{\mathbb{R}} \psi(\xi^T K(F(x)))d\mu$$

Dual representation

If ψ is derivable and there exists a solution ξ^* of the dual problem which is an interior point of $\{\xi \in \mathbb{R}^l \text{ s.t. } \int_{\mathbb{R}} \psi(<\xi, K(F(x))>)d\mu < \infty\}$, then ξ^* is the unique maximum checking :

$$\int \psi'({\xi^*}^T K \circ F(x)) K \circ F(x) d\mu = f(\theta)$$

and $\theta \mapsto \xi^*(\theta)$ is continuous.

Dual representation for χ^2-divergence

With the χ^2-divergence $\varphi(x) = \frac{(x-1)^2}{2}$, $\psi(t) = \frac{1}{2}t^2 + t$
The solution ξ_1^* of the dual problem is

$$\xi^* = \Omega^{-1}\left(f(\theta) - \int K(F(x))d\mu\right)$$

with

$$\Omega = \int K(F(x))K(F(x))^T d\mu$$

And the estimator is then

$$\hat{\theta}_n = \arg\inf_{\theta\in\Theta}\left(f(\theta) - \int K(F_n(x))d\mu\right)\Omega_n^{-1}\left(f(\theta) - \int K(F_n(x))d\mu\right)$$

with $\Omega_n = \int K(F_n(x))K(F_n(x))^T d\mu$

Asymptotic properties of the estimators under the model

Theorem

Let $X_1, \ldots, X_n$ be random samples coming from the same distribution F_0. Let suppose that there exists θ_0 such that

- *$F_0 \in M_{\theta_0}$, θ_0 is the unique solution of the equation $f(\theta) = f(\theta_0)$*
- *f is continuous and Θ is compact*
- *the matrix $\Omega = \int K(F_0(x))K(F_0(x))^T dx$ is non singular.*

Then with probability approaching one,

$$\hat{\theta}_n \xrightarrow{p} \theta_0$$

Asymptotic properties of the estimators under the model

Asymptotic normality of the estimator :

Theorem

Let define

- $J_0 = J_f(\theta_0)$ *be the Jacobian of* f *with respect to* θ *in* θ_0
- $M = (J_0^T \Omega^{-1} J_0)^{-1}$, $H = M J_0^T \Omega^{-1}$,
  $P = \Omega^{-1} - \Omega^{-1} J_0 M J_0^T \Omega^{-1}$
- $\Sigma = \int\int [F_0(\min(x,y)) - F_0(x)F_0(y)] K'(F_0(x)).K'(F_0(y)) dxdy$

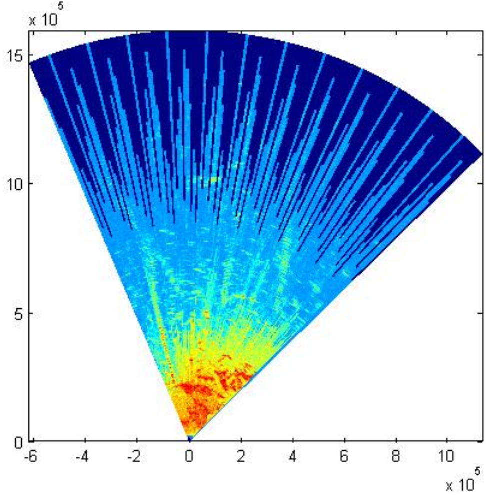
Then,

$$\sqrt{n} \begin{pmatrix} \hat{\theta}_n - \theta_0 \\ \hat{\xi}_n \end{pmatrix} \xrightarrow{d} N(0, diag(H \Sigma H^T, P \Sigma P^T))$$

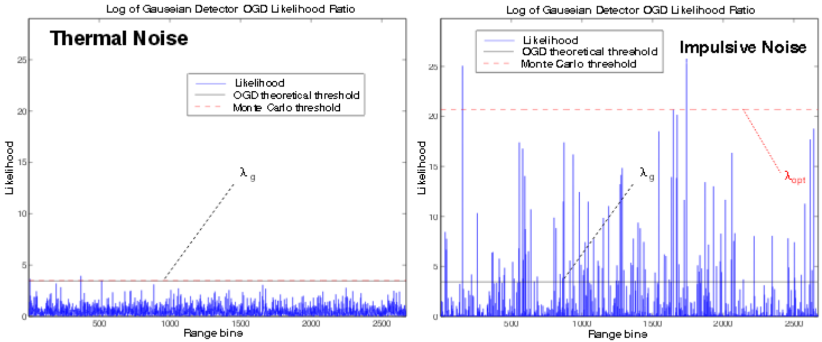
Further work

- Same asymptotic properties under misspecification
- Extension to multivariate case

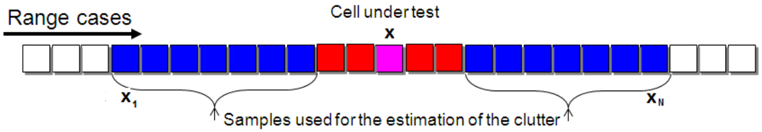
Radar power data



Thermal noise vs impulsive noise



Processing chain for adaptive detection



	H_0 (no target)	H_1 (target)
Decision D=0 : no target	Noise $(1 - P_{fa})$	Miss $(1 - P_{det})$
Decision D=1 : target	False alarm (P_{fa})	Detection (P_{det})

Constraint : keep the false alarm constant with robustness to

- misspecification
- other targets

Aim : estimation of H_0 distribution