Extremal behavior of stochastic risk processes

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Outline

The Kinetic Dietary Exposure Model (KDEM)

Represent the evolution of a contaminant in the human body through time.
The quantity of contaminant is driven by the Markov chain by

$$S_{n+1} = e^{-\omega \Delta T_n} S_n + W_{n+1}, \quad n \geq 0. \tag{1.1}$$

We suppose that

- $(W_n)_{n\geq 0}$ *i.i.d.* $RV_{-\alpha}, \alpha > 0$ r.v.'s, with distribution F, called "claims" or "intakes".

It means that

$$\lim_{t\to\infty} \frac{\overline{F}(tx)}{\overline{F}(t)} = x^{-\alpha}, \quad x > 0.$$

The Kinetic Dietary Exposure Model (KDEM)

- $(\Delta T_n)_{n\geq 0}$ is a **renewal sequence** called "inter-arrivals".

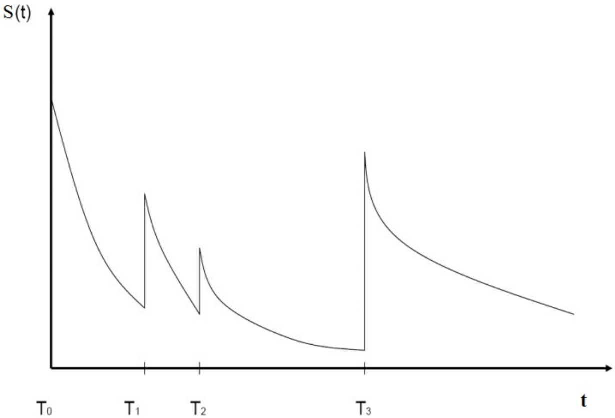
They generate the "claim instants", defined by

$$T_i = \sum_{k\leq i} \Delta T_k, \quad i \geq 0.$$

By convention, $T_0 = 0$.

- ω is an elimination parameter (usually chosen as log(2)/Half Life).

KDEM



Kinetic Dietary Exposure Model (KDEM)

The KDEM

Main aim: to understand its extremal behavior $\Rightarrow$ **to study its maxima**. Why?

- To develop risk indicator.
- To answer to revelent safety dietary questions.

Assume $S(0) = 0$. The embedded chain of

$$S(t) = \sum_{i=0}^{N(t)} W_i e^{-\omega(t-T_i)} \tag{1.2}$$

is the autoregressive process (1.1), with N a **renewal process** such that for all $t > 0$,

$$N(t) = \# \{i : T_i \leq t\} .$$

Remarks

Remark

(1.2) *is a particular kind of Shot Noise Process*

Remark

The study of the extremal behavior of the continuous time process (1.2) *leads to the study of the maxima of the discrete time process* (1.1).

Shot Noise Processes

We will study a more general process on the form

$$S(t) = \sum_{i=0}^{N(t)} W_i h_i(t - T_i), \tag{1.3}$$

where h is usually called "shot" or "response" function.
$\Rightarrow$ permits to modify the impacts of the claims.

Examples:

- Interest or discount factor.
- Delayed or reported parameter (IBNR).

More examples

The DKDEM defined for $t > 0$ by

$$S^{(1)}(t) = \sum_{i=1}^{N(t)} W_i \times a^{-1}(t - T_i + \tau)^k e^{-\omega(t-T_i)}, \qquad (1.4)$$

with $a > 0$, $\tau \geq 0$, $k \geq 0$ respectively normalization, centering, and delay parameters.

Then, for $t > 0$ and $i > 0$,

$$h_i(t) = a^{-1}(t + \tau)^k e^{-\omega t}.$$

More examples

The IESM defined for $t > 0$ by

$$S^{(2)}(t) = \sum_{i=1}^{N(t)} W_i \mathbb{I}_{[0,\xi_i]}(t - T_i), \tag{1.5}$$

with $(\xi_i)_i$ a i.i.d. sequence of positive r.v's.

The KDEM-RREE defined for all $t > 0$ by

$$S^{(3)}(t) = \sum_{i=1}^{N(t)} W_i \times e^{-\omega_i (t - T_i)}, \tag{1.6}$$

where $(\omega_i)_i$ is a sequence of i.i.d. r.v.'s.

The Tail Behahior

The process (1.2) **is not stationary** $\to$ We study a stationary version. To this end, assume

(C1) The sequence

$$\cdots < \tilde{T}_{-2} < \tilde{T}_{-1} < \tilde{T}_0 \leq 0 < \tilde{T}_1 < \tilde{T}_2 < \cdots$$

forms the stationary renewal version of $(T_i)_{i \in \mathbb{N}}$.

(C2) The survival function of the intakes W is $RV_{-\alpha}$.

(C3) For all $i \geq 0$, the shock functions $h_i : \mathbb{R}^+ \to \mathbb{R}^+$ are positive, stationary and independent of the sequence $(\tilde{T}_i)_{\in \mathbb{N}}$. Moreover,

$$\int_0^\infty \mathbb{E}[h_1(t)^{1+\alpha}]dt < \infty.$$

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The Tail Behahior

One may define the stationary process associated with S as

$$X(t) = \sum_{i=-\infty}^{\infty} W_i \times h_i(t - \tilde{T}_i), \qquad t \in \mathbb{R}. \tag{2.1}$$

Observe that, for all $t \geq 0$,

$$X(t) \stackrel{d}{=} X(0) \stackrel{d}{=} \sum_{i=1}^{\infty} W_i h_i(\tilde{T}_i),$$

Proposition (Tail behavior)

Under (C1)-(C3), we have

$$\lim_{x\to\infty} \frac{\mathbb{P}(X(0) > x)}{\overline{F}(x)} = \lambda \int_0^{\infty} \mathbb{E}[h_1^{\alpha}(t)]dt. \tag{2.2}$$

The extremal index

We describe the extremal behavior of (1.2) by its extremal index.

Definition (The extremal index)

Let $(Z_i)_{i\in\mathbb{Z}}$ be a stationary sequence. The sequence $(Z_i)_{i\in\mathbb{Z}}$ has an extremal index denoted by $\theta \in [0,1]$ if for all $\tau > 0$ and all sequence $u_n(\tau)$ such that

$$\lim_{n\to\infty} \mathbb{P}(Z_1 \geq u_n(\tau)) = \tau,$$

it holds that

$$\lim_{n\to\infty} \mathbb{P}\left(\max_{i=1,\dots,n} Z_i \leq u_n(\tau)\right) = e^{-\theta\tau}.$$

The extremal index

Work on the embedded chain of $\{S(t)\}_{t\geq 0}$ defined by

$$S(T_k) = \sum_{i=1}^{k} W_i h_i(T_k - T_i), \qquad k > 0. \tag{2.3}$$

Issue of non stationarity $\Rightarrow$ Study a stationary version of (2.3), denoted $(S_k)_{k\in\mathbb{Z}}$ and defined by

$$S_k = \sum_{i=-\infty}^{k} W_i h_i(T_k - T_i). \tag{2.4}$$

Assume (D1) that the random functions h_i satisfy

$$\sum_{i=0}^{\infty} \mathbb{E}[h_1(T_i)^{1+\alpha}] < \infty.$$

The extremal index

Theorem

If (C2) and (D1) are satisfied, then we have

$$\lim_{x\to\infty} \frac{\mathbb{P}(S_0 > x)}{\overline{F}(x)} = \sum_{i=1}^{\infty} \mathbb{E}\left[h_1^\alpha(T_i)\right]. \tag{2.5}$$

Moreover, its extremal index is given by

$$\theta = \frac{\mathbb{E}\left[\max_{i\geq 0} h_i^\alpha(T_i)\right]}{\sum_{i=0}^{\infty} \mathbb{E}[h_i^\alpha(T_i)]}. \tag{2.6}$$

The KDEM-RREE

Consider

- $T = \infty$
- $\omega \equiv (\omega_i)_{i \in \mathbb{N}}$ i.i.d. ≥ 0 r.v's s.t. $\mathbb{E}[\omega^{-1}] < \infty$
- The intake instants $T_0, T_1, \ldots$ are Poisson distributed with intensity λ

Proposition

Let the assumptions (D1)-(D2) hold, set $h_i(t) = e^{-\omega_i t}$. Then, the extremal index is given by

$$\theta = \frac{\alpha}{\alpha + \lambda \mathbb{E}[\omega^{-1}]}$$

Issues

- Explicit formula in several cases especially in a Poisson setup (OSPP)
- Troubles when the h_i are not monotonic to compute $\mathbb{E}\left[\max_{i\geq 0} h_i^{\alpha}(T_i)\right]$
- Recours to simulation methods such as Monte Carlo

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